Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 12, 2004*
Accepted: *May 12, 2004*

Self-dual super–Yang-Mills as a string theory in (x, θ) space

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Abstract: Different string theories in twistor space have recently been proposed for describing $\mathcal{N} = 4$ super-Yang-Mills. In this paper, a string theory in (x, θ) space is constructed for self-dual $\mathcal{N} = 4$ super-Yang-Mills. It is hoped that these results will be useful for understanding the twistor-string proposals and their possible relation with the pure spinor formalism of the $d = 10$ superstring.

Keywords: Superstrings and Heterotic Strings, Extended Supersymmetry.

JHEP05(2004)034

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1. Motivation

At the present time, the only quantizable description of the superstring in an $AdS_5 \times S^5$ background uses the pure spinor formalism of the superstring [1, 2].[1] When the AdS radius goes to zero, this string theory is conjectured to describe perturbative $\mathcal{N} = 4$ super–Yang-Mills [5]. But because the string theory becomes a strongly coupled sigma model when the AdS radius goes to zero, it has not yet been possible to test this conjecture directly using the pure spinor formalism. Perhaps the integrable currents found in [6] will be useful for connecting the large and small radius limits of the string theory.

Although one cannot yet directly study the small radius limit of the closed superstring in an $AdS_5 \times S^5$ background, it might be possible to guess the structure of the string theory using the knowledge that it should describe $\mathcal{N} = 4$ $d = 4$ super–Yang-Mills theory. By analogy with the Gopakumar-Vafa duality [7] where closed strings on the conifold are dual to open strings on S^3 for Chern-Simons theory, one might expect this $\mathcal{N} = 4$ super–Yang-Mills theory to be described by an open string theory.

As was described in unpublished work at Strings 2003 [8], one can construct an open string theory for self-dual $\mathcal{N} = 4$ $d = 4$ super–Yang-Mills which closely resembles the $d = 10$ pure spinor formalism. Unlike the recent string theories constructed in twistor space [9],

[1] Although one can semi-classically describe the superstring in an $AdS_5 \times S^5$ background using the covariant Green-Schwarz formalism [3], this description is not fully quantum. Note that even in a flat background, the covariant Green-Schwarz formalism has not yet been used to compute superstring scattering amplitudes. Although the light-cone Green-Schwarz formalism can be quantized in a flat background, there are subtleties with quantizing the light-cone gauge formalism in an AdS background [4].

this string theory is constructed in (x, θ) space and only describes the self-dual super–Yang-Mills interactions. However, this string theory contains bosonic spinor worldsheet variables which might be related to the recent twistor-string constructions.

In section 2, the $N = 2$ string in $d = (2, 2)$ target-space is reviewed using the $N = 4$ topological description which involves constant bosonic spinors. In section 3, it is shown that these bosonic spinors can be treated as worldsheet variables if the $d = 4$ target space is supersymmetrized to an $\mathcal{N} = 4$ $d = 4$ target superspace. In section 4, this string theory is related to a dimensional reduction of the $d = 10$ pure spinor formalism of the superstring. In sections 5 and 6, vertex operators and scattering amplitudes are computed and shown to correspond to self-dual $\mathcal{N} = 4$ $d = 4$ super–Yang-Mills. And section 7 contains conclusions and speculations.

2. Review of open $N = 2$ string

Although no open string theory was known to describe $\mathcal{N} = 4$ $d = 4$ super–Yang-Mills theory at the time of the Strings 2003 conference, there is an open string theory which describes $\mathcal{N} = 0$ self-dual Yang-Mills theory in $d = (2, 2)$. This string theory is described by the worldsheet action [10, 11, 12, 13]

$$S = \int d^2z \left(\frac{1}{2}\partial x^{a\dot{a}} \bar{\partial} x_{a\dot{a}} + \eta^{\dot{a}} \bar{\partial} \psi_{\dot{a}} + \bar{\eta}^{\dot{a}} \partial \bar{\psi}_{\dot{a}}\right) \tag{2.1}$$

and the $\hat{c} = 2$ $N = 2$ superconformal generators

$$\begin{aligned} T &= \frac{1}{2}\partial x^{a\dot{a}} \partial x_{a\dot{a}} + \frac{1}{2}(\eta^{\dot{a}} \partial \psi_{\dot{a}} - \psi^{\dot{a}} \partial \eta_{\dot{a}})\,, \\ G^+ &= \psi^{\dot{a}} \partial x_{1\dot{a}}\,, \qquad G^- = \eta^{\dot{a}} \partial x_{2\dot{a}}\,, \\ J &= \eta^{\dot{a}} \psi_{\dot{a}}\,, \end{aligned} \tag{2.2}$$

where a and $\dot{a}$ are independent two-component $\mathrm{SL}(2, \mathbb{R})$ indices and $(\eta^{\dot{a}}, \psi^{\dot{a}})$ and $(\bar{\eta}^{\dot{a}}, \bar{\psi}^{\dot{a}})$ are left and right-moving fermions. Physical states in $N = 2$ string theory are defined as dimension-zero U(1)-neutral primary fields with respect to these generators, and the only physical state is

$$\phi(x(z)) = e^{ik^{a\dot{a}} x_{a\dot{a}}(z)}$$

where $k^{a\dot{a}} k_{a\dot{a}} = 0$.

Computing the tree-level scattering amplitudes of these physical states, one finds that $\phi(x)$ satisfies the classical equations of motion [11]

$$\epsilon^{\dot{a}\dot{b}} \partial_{1\dot{a}} (e^{-\phi} \partial_{2\dot{b}} e^{\phi}) = 0\,, \tag{2.3}$$

which describes a self-dual Yang-Mills field in the Yang description. Note that in the Yang description, the self-dual gauge field $A_{a\dot{a}}$ is defined non-covariantly in terms of ϕ as

$$A_{1\dot{a}} = 0\,, \qquad A_{2\dot{a}} = e^{-\phi} \partial_{2\dot{a}} e^{\phi}\,, \tag{2.4}$$

which implies that the anti-self-dual field strength

$$F_{ab} = \epsilon^{\dot{a}\dot{b}}(\partial_{a\dot{a}}A_{b\dot{b}} - \partial_{b\dot{b}}A_{a\dot{a}} + [A_{a\dot{a}}, A_{b\dot{b}}]) \tag{2.5}$$

vanishes when ϕ satisfies (2.3).

Although the equation $F_{ab} = 0$ is manifestly Lorentz covariant, one cannot obtain (2.3) from a covariant action constructed from ϕ. A related problem is that loop amplitudes for $\phi(x)$ computed using the $N = 2$ string theory do not correspond to loop amplitudes of self-dual Yang-Mills theory [13].

To make Lorentz invariance manifest, it is useful to construct a set of "small" $N = 4$ superconformal generators out of the worldsheet variables as [14, 15]

$$\begin{aligned} T &= \frac{1}{2}\partial x^{a\dot{a}}\partial x_{a\dot{a}} + \frac{1}{2}(\eta^{\dot{a}}\partial\psi_{\dot{a}} - \psi^{\dot{a}}\partial\eta_{\dot{a}})\,, \\ G^+ &= \psi^{\dot{a}}\partial x_{1\dot{a}}\,, \qquad \tilde{G}^+ = \psi^{\dot{a}}\partial x_{2\dot{a}}\,, \qquad G^- = \eta^{\dot{a}}\partial x_{2\dot{a}}\,, \qquad \tilde{G}^- = -\eta^{\dot{a}}\partial x_{1\dot{a}}\,, \\ J^{++} &= \psi^{\dot{a}}\psi_{\dot{a}}\,, \qquad J = \psi^{\dot{a}}\eta_{\dot{a}}\,, \qquad J^{--} = \eta^{\dot{a}}\eta_{\dot{a}}\,, \end{aligned} \tag{2.6}$$

where "small" means there is an $\mathrm{SL}(2,\mathbb{R})$ set of dimension-one currents as opposed to an $\mathrm{SO}(2,2)$ set.

As shown in [15], one can describe this $N = 2$ string as an "$N = 4$ topological string" by twisting $\psi^{\dot{a}}$ to have spin zero and $\eta^{\dot{a}}$ to have spin one, so that $(G^+, \tilde{G}^+)$ have spin one and $(G^-, \tilde{G}^-)$ have spin two. Since the stress tensor has zero central charge after twisting, there is no need to introduce super-Virasoro ghosts. Physical states are described by the cohomology of the BRST operator

$$Q = \lambda^1 \int dz G^+ + \lambda^2 \int dz \tilde{G}^+ = \lambda^a \int dz \psi^{\dot{a}}\partial x_{a\dot{a}} \tag{2.7}$$

where (λ^1, λ^2) are constants which parameterize the coset $\mathrm{SL}(2,\mathbb{R})/GL(1)$. This is a natural $N = 4$ generalization of the $N = 2$ topological string where physical states are described by the cohomology of the BRST operator $Q = \int dz G^+$.

One finds that the unique state in the cohomology of (2.7) is

$$V = \lambda^a \psi^{\dot{a}} A_{a\dot{a}}(x)$$

where $\epsilon^{\dot{a}\dot{b}}(\partial_{a\dot{a}}A_{b\dot{b}} - \partial_{b\dot{b}}A_{a\dot{a}}) = 0$. So the vertex operator V describes self-dual Yang-Mills in a Lorentz-covariant manner. However, using this $N = 4$ topological description, tree amplitudes involving V are not manifestly Lorentz covariant and loop amplitudes do not correspond to those of self-dual Yang-Mills theory.

3. String theory for self-dual super–Yang-Mills

The lack of manifest Lorentz covariance is related to the fact that λ^a is a constant parameter and not a worldsheet variable. So to restore Lorentz invariance,[2] one should treat λ^a as a

[2] Another approach to restoring Lorentz invariance is to introduce Chan-Paton factors which describe different spacetime helicities and imply spacetime supersymmetry [16, 17]. See also [18] for other approaches to restoring Lorentz invariance and supersymmetry.

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worldsheet variable and define $Q = \int dz \lambda^a \psi^{\dot a} \partial x_{a\dot a}$. But since

$$G^+(y)\tilde{G}^+(z) \to \frac{J^{++}(y) + J^{++}(z)}{2(y-z)^2},$$

Q^2 is nonvanishing when λ^a is not constant and one needs to modify the BRST charge. One possible modification is

$$\begin{aligned} Q &= \int dz(\lambda^1 G^+ + \lambda^2 \tilde{G}^+ + eJ^{++} + hJ + \text{ghost terms}) \\ &= \int dz(\lambda^a \psi^{\dot a} \partial x_{a\dot a} + e\psi^{\dot a}\psi_{\dot a} + f\lambda^a \partial\lambda_a + h(-\lambda^a w_a + \psi^{\dot a}\eta_{\dot a} + 2fe)) \end{aligned} \tag{3.1}$$

where (w_a, f, g) are the conjugate antighosts for (λ^a, e, h). The Lorentz covariant worldsheet action for these fields is

$$S = \int d^2z \left(\frac{1}{2}\partial x^{a\dot a}\bar\partial x_{a\dot a} + \eta^{\dot a}\bar\partial\psi_{\dot a} + w^a\bar\partial\lambda_a + e\bar\partial f + g\bar\partial h + \bar\eta^{\dot a}\partial\bar\psi_{\dot a} + \bar w^a \partial\bar\lambda_a + \bar e\partial\bar f + \bar g\partial\bar h \right), \tag{3.2}$$

and the desired effect of the fermionic ghosts (e, f) and (g, h) is to cancel the contribution of the bosonic $(w_\alpha, \lambda^\alpha)$ ghosts to the BRST cohomology. This is plausible since the h ghost couples in Q to the constraint $J = -\lambda^a w_a + \psi^{\dot a}\eta_{\dot a} + 2fe$ which generates projective transformations of λ^a, and the f ghost couples to the constraint $\lambda^a \partial\lambda_a$ which fixes λ^2/λ^1 to be constant. However, since $J(y)J(z) \to 4(y-z)^{-2}$, $Q^2 = 4\int dz(h\partial h)$ and the theory is anomalous. Fortunately, as will now be shown, this anomaly can be cured by replacing $\mathcal{N} = 0$ self-dual Yang-Mills with $\mathcal{N} = 4$ self-dual super–Yang-Mills.

As shown by Siegel in [16, 19], one can describe self-dual super–Yang-Mills in superspace by extending the $x^{a\dot a}$ spacetime variables to $(x^{a\dot a}, \theta^{aj})$ superspace variables where $j = 1$ to $\mathcal{N}$. Note that one does not need to include $\bar\theta^{\dot a\bar j}$ variables since the antichiral superspace derivatives $\bar\nabla_{\dot a\bar j}$ satisfy the trivial (anti)commutation relations

$$\{\bar\nabla_{\dot a\bar j}, \bar\nabla_{\dot b\bar k}\} = [\bar\nabla_{\dot b\bar k}, \nabla_{a\dot a}] = 0$$

when the theory is self-dual. It will be convenient to combine the $(x^{a\dot a}, \theta^{aj})$ variables into a superspace variable

$$Y^{aJ} = (x^{a\dot a}, \theta^{aj}) \tag{3.3}$$

where $J = (j, \dot a)$ is an $\mathrm{OSp}(\mathcal{N}|2)$ vector index. The variable Y^{aJ} is defined to transform covariantly under $\mathrm{SL}(2,\mathbb{R})$ rotations of the a index and under $\mathrm{OSp}(\mathcal{N}|2)$ rotations of the J index. Just as $(\eta^{\dot a}, \psi^{\dot a})$ are the fermionic worldsheet superpartners of $x^{a\dot a}$, one can define (b^j, c^j) to be the bosonic worldsheet superpartners of θ^{aj}. These worldsheet superpartners of Y^{aJ} can be described in $\mathrm{OSp}(\mathcal{N}, 2)$ notation as

$$C^J = (\psi^{\dot a}, c^j), \qquad B^J = (\eta^{\dot a}, b^j). \tag{3.4}$$

The open string for self-dual super–Yang-Mills will be defined by the worldsheet action

$$S = \int d^2z \left(\frac{1}{2}\partial Y^{aJ}\bar\partial Y_{aJ} + B^J\bar\partial C_J + w^a\bar\partial\lambda_a + e\bar\partial f + g\bar\partial h + \bar B^J\partial\bar C_J + \bar w^a\partial\bar\lambda_a + \bar e\partial\bar f + \bar g\partial\bar h \right), \tag{3.5}$$

the stress tensor

$$T = \frac{1}{2}\partial Y^{aJ}\partial Y_{aJ} + B^J \partial C_J + w^a \partial\lambda_a + e\partial f + g\partial h\,, \tag{3.6}$$

and the BRST operator

$$Q = \int dz(\lambda^a C^J \partial Y_{aJ} + eC^J C_J + f\lambda^a\partial\lambda_a + h(-\lambda^a w_a + C^J B_J + 2fe))\,, \tag{3.7}$$

where the J index is raised and lowered using the OSp($\mathcal{N}|2$) metric which is symmetric/antisymmetric when $J = j/\dot{a}$. Note that the action is invariant under only an SL(2) $\times$ OSp($\mathcal{N}|2$) subgroup of the full superconformal group PSU($2,2|4$). Also note that the kinetic term is $\frac{1}{2}\int d^2z \partial Y^{aJ}\bar{\partial}Y_{aJ} = \frac{1}{2}\int d^2z(\partial x^{a\dot{a}}\bar{\partial}x_{a\dot{a}} + \partial\theta^{aj}\bar{\partial}\theta_{aj})$, so θ^{aj} satisfies the non-holomorphic OPE's

$$\theta^{aj}(y,\bar{y})\theta^{bk}(z,\bar{z}) \to \epsilon^{ab}\delta^{jk}\log|y-z|^2\,.$$

One can easily check that $J = -\lambda^a w_a + C^J B_J + 2fe$ now satisfies the OPE $J(y)J(z) \to (4-\mathcal{N})(y-z)^{-2}$, so the theory is not anomalous when $\mathcal{N} = 4$.

4. Comparison with pure spinor formalism

To show that the theory described by (3.5), (3.6) and (3.7) describes $\mathcal{N} = 4$ self-dual super–Yang-Mills, it is useful to first review the pure spinor formalism of the superstring in a flat $d = 10$ background [1]. In a flat $d = 10$ background, the pure spinor formalism for the superstring is described by the worldsheet action

$$S = \int d^2z\left(\frac{1}{2}\partial x^m\bar{\partial}x_m + p_\alpha\bar{\partial}\theta^\alpha + w_\alpha\bar{\partial}\lambda^\alpha + \bar{p}_\alpha\partial\bar{\theta}^\alpha + \bar{w}_\alpha\partial\bar{\lambda}^\alpha\right), \tag{4.1}$$

the stress tensor

$$T = \frac{1}{2}\partial x^m\partial x_m + p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha\,, \tag{4.2}$$

and the BRST operator

$$Q = \int dz\lambda^\alpha d_\alpha \tag{4.3}$$

where $m = 0$ to 9, $\alpha = 1$ to 16,

$$d_\alpha = p_\alpha + \gamma^m_{\alpha\beta}\partial x_m\theta^\beta - \frac{1}{2}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha$$

is the Green-Schwarz constraint, and λ^α is a pure spinor satisfying

$$\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0\,.$$

Massless $d = 10$ super–Yang-Mills states of the open superstring are described by the unintegrated vertex operator

$$V = \lambda^\alpha A_\alpha(x,\theta) \tag{4.4}$$

and by the integrated vertex operator

$$U = \partial\theta^\alpha A_\alpha(x,\theta) + \pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + \frac{1}{2}(w\gamma^{mn}\lambda)F_{mn}(x,\theta) \tag{4.5}$$

where $\pi^m = \partial x^m + \theta\gamma^m\partial\theta$ is the supersymmetric momentum, (A_α, A_m) are the spinor and vector gauge superfields satisfying

$$D_{(\alpha}A_{\beta)} = \gamma^m_{\alpha\beta}A_m\,, \tag{4.6}$$

and W^α and F_{mn} are the spinor and vector field strength superfields. Although there is no b ghost in this formalism, the integrated and unintegrated vertex operators are related to each other by $QU = \partial V$.

N-point tree amplitudes are computed in the pure spinor formalism by the correlation function

$$A = \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U_4(z_4)\cdots\int dz_N U_N(z_N)\rangle \tag{4.7}$$

on a disk. Since the worldsheet action is quadratic, the only nontrivial aspect of computing this correlation function is the normalization of worldsheet zero modes. The correct normalization is

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1\,, \tag{4.8}$$

which is BRST invariant since $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ is the unique state of ghost-number three in the BRST cohomology. Recall that in bosonic string theory, $c\partial c\partial^2 c$ is the unique state of ghost-number three in the BRST cohomology and $\langle c\partial c\partial^2 c\rangle = 1$ is the correct normalization for zero modes on a disk. Furthermore, one can check that the field theory action

$$\mathcal{S} = \langle VQV + V^3\rangle \tag{4.9}$$

reproduces the standard $d = 10$ super–Yang-Mills action if one uses the normalization of (4.8) and the string field $V = \lambda^\alpha A_\alpha(x,\theta)$.

Since $d = 10$ super–Yang-Mills dimensionally reduces to $\mathcal{N} = 4$ $d = 4$ super–Yang-Mills (which has the same spectrum as $\mathcal{N} = 4$ self-dual super–Yang-Mills), it is natural to expect that the open string theory for $\mathcal{N} = 4$ self-dual super–Yang-Mills is related to a dimensional reduction of the $d = 10$ pure spinor formalism. This relationship can be made more explicit by first decomposing the $d = 10$ spinors λ^α and θ^α for $\alpha = 1$ to 16 into the $d = 4$ SU(4) spinors $(\lambda^{aj}, \bar\lambda^{\dot a\bar j})$ and $(\theta^{aj}, \bar\theta^{\dot a\bar j})$ where $(a, \dot a) = 1$ to 2 and $(j, \bar j) = 1$ to 4. If one then sets[3]

$$\lambda^{aj} = \lambda^a c^j, \quad \bar\lambda^{\dot a\bar j} = 0, \tag{4.10}$$

the pure spinor constraint is satisfied and the BRST operator of (4.3) becomes

$$Q = \int dz\lambda^{aj}d_{aj} = \int dz\lambda^a c^j\left(p_{aj} + \bar\theta^{\dot a\bar j}\partial x_{a\dot a} + \cdots\right), \tag{4.11}$$

where $\cdots$ includes terms quadratic in $\bar\theta^{\dot a\bar j}$. After defining

$$\psi^{\dot a} = c^j\bar\theta^{\dot a\bar j} \tag{4.12}$$

[3]A similar decomposition of pure spinors has recently been used in the interesting paper of [20] to obtain $d = 4$ harmonic superspaces with $\mathcal{N} = 2, 3$ or 4 supersymmetry.

and recognizing that $\partial\theta^{aj}$ in the second-order action of (3.5) plays the role of p_{aj} in the first-order action of (4.1), one can reexpress the BRST operator of (4.11) as

$$Q = \int dz \lambda^a \left(c^j \partial\theta_{aj} + \psi^{\dot a} \partial x_{a\dot a} + \cdots \right) , \tag{4.13}$$

which closely resembles (3.7). In the same sense that the pure spinor formalism can be understood as covariant quantization of the Green-Schwarz superstring, the self-dual formalism described by (3.7) can be understood as covariant quantization of the self-dual Green-Schwarz superstring proposed by Siegel in [19].

5. Self-dual super–Yang-Mills vertex operators

By analyzing the cohomology of the BRST operator of (3.7), one finds a similar structure to the massless cohomology in the pure spinor formalism. At ghost number one (where the ghost-number charge is defined as $\int dz(\lambda^a w_a + ef - gh)$), the unique state in the cohomology is described by the unintegrated vertex operator

$$V = \lambda^a C^J A_{aJ}(Y) = \lambda^a (\psi^{\dot a} A_{a\dot a}(x,\theta) + c^j A_{aj}(x,\theta)) , \tag{5.1}$$

where A_{aJ} satisfies

$$\partial_{aJ} A_{bK} - (-1)^{s(K)} \partial_{bK} A_{aJ} = \epsilon_{ab} F_{JK} \tag{5.2}$$

and $s(K) = 1/0$ when K is $j/\dot a$. Although there is no b ghost, one can define the integrated vertex operator by $QU = \partial V$ and one finds that

$$U = \partial Y^{aJ} A_{aJ}(Y) + B^J C^K F_{JK}(Y) . \tag{5.3}$$

Note the close resemblance of equations (5.1) - (5.3) to equations (4.4) - (4.6) in the pure spinor formalism.

To see that these vertex operators describe self-dual super–Yang-Mills, note that (5.2) is the supersymmetrization of the $\mathcal{N} = 0$ self-dual Yang-Mills relation $[\nabla_{a\dot a}, \nabla_{b\dot b}] = \epsilon_{ab} F_{\dot a\dot b}$. In components, A_{aJ} can be gauge-fixed to

$$\begin{aligned} A_{a\dot a} &= a_{a\dot a} + \theta^j_a \bar\xi_{\dot a j} + \theta^j_a \theta^{bk} \partial_{b\dot a} \phi^{lm} \epsilon_{jklm} + \theta^j_a \theta^{bk} \theta^{cl} \partial_{b\dot a} \xi^m_c \epsilon_{jklm} + (\theta^4)^{bcd}_a \partial_{b\dot a} G_{cd} , \\ A_{aj} &= (\theta^k_a \phi^{lm} + \theta^k_a \theta^l_b \xi^{bm} + \theta^k_a \theta^l_b \theta^m_c G^{bc}) \epsilon_{jklm} , \end{aligned} \tag{5.4}$$

where $a_{a\dot a}$ is the self-dual gauge field, ξ^{bm} and $\bar\xi_{\dot a j}$ are the on-shell gluinos, ϕ^{lm} are the six on-shell scalars, and G^{bc} is the anti-self-dual field strength. The $\mathcal{N} = 4$ self-dual super–Yang-Mills action for these component fields is

$$S = \mathrm{Tr} \int d^4x \left[\left(\partial_{a\dot a} a^{\dot a}_b + \partial_{b\dot a} a^{\dot a}_a + [a_{a\dot a}, a^{\dot a}_b] \right) G^{ab} + \xi^{bm} \nabla_{b\dot b} \bar\xi^{\dot b}_m + \nabla_{a\dot a} \phi^{jk} \nabla^{a\dot a} \phi^{lm} \epsilon_{jklm} \right] , \tag{5.5}$$

where $\nabla_{a\dot a} = \partial_{a\dot a} + a_{a\dot a}$. Note that this action has the same spectrum as ordinary $\mathcal{N} = 4$ super–Yang-Mills but has different interactions [16].

Unlike the pure spinor formalism, there are no massive states in the cohomology of Q of (3.7). As in the $\mathcal{N} = 0$ self-dual string, this is because the constraints $C^J \partial Y_{aJ}$ in the

BRST operator imply that all states in the cohomology are independent of the nonzero worldsheet modes of Y_{aJ} and (C^J, B_J). Furthermore, states are independent of nonzero modes of $(\lambda^a, w_a, e, f, g, h)$ because of the terms $f\lambda^a\partial\lambda_a$ and $h(-\lambda^a w_a + C^J B_J + 2ef)$ in Q. The absence of physical massive states will be verified below by showing there are no massive poles in the scattering amplitudes.

6. Self-dual super–Yang-Mills tree amplitudes

To compute N-point tree amplitudes in this formalism, one uses the same prescription as before that

$$A = \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U_4(z_4)\cdots\int dz_N U_N(z_N)\rangle\,. \tag{6.1}$$

Since the unique state at ghost-number three in the cohomology is

$$\lambda^a\lambda^b\lambda^c\psi^{\dot a}\psi_{\dot a}\theta^j_a\theta^k_b\theta^l_c c^m\epsilon_{jklm}\,, \tag{6.2}$$

one should define normalization of the zero modes by

$$\langle\lambda^a\lambda^b\lambda^c\psi^{\dot a}\psi_{\dot a}\theta^j_a\theta^k_b\theta^l_c c^m\epsilon_{jklm}\rangle = 1\,. \tag{6.3}$$

Although this normalization appears strange, it is obtained from the $d = 10$ normalization $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$ by performing the dimensional reduction of (4.10) and (4.12) where one sets $\lambda^{aj} = \lambda^a c^j$, $\bar\lambda^{\dot a\bar j} = 0$, and $\psi^{\dot a} = c^j\bar\theta^{\dot a\bar j}$.

One can easily check that this tree amplitude prescription reproduces the desired self-dual super–Yang-Mills amplitudes, which vanish unless the external momenta are chosen to lie in the same "self-dual plane". To see this, note that when the self-dual gauge superfield $A_{aJ}(Y)$ is onshell, it can be expressed as

$$\begin{aligned} A_{a\dot a}(Y) &= \int d^4\kappa\,\bar\pi_{\dot a}e^{i\pi^b(\bar\pi^{\dot b}x_{b\dot b}+\kappa^j\theta_{bj})}\Phi_a(\pi,\bar\pi,\kappa)\,,\\ A_{aj}(Y) &= \int d^4\kappa\,\kappa_j\,e^{i\pi^b(\bar\pi^{\dot b}x_{b\dot b}+\kappa^j\theta_{bj})}\Phi_a(\pi,\bar\pi,\kappa)\,, \end{aligned} \tag{6.4}$$

where $\Phi_a(\pi,\bar\pi,\kappa)$ is unconstrained, $p_{a\dot a} = \pi_a\bar\pi_{\dot a}$ is the momentum, and κ_j is the fermionic conjugate momentum to $\pi^a\theta_{aj}$. These superfields can be expressed in OSp(4|2)-covariant notation as

$$A_{aJ}(Y) = \int d^4\kappa\,\bar\Pi_J e^{i\pi^b\bar\Pi^J Y_{bJ}}\Phi_a(\pi,\bar\Pi)\,, \tag{6.5}$$

where $\bar\Pi^{\dot a} = \bar\pi^{\dot a}$ and $\bar\Pi^j = \kappa^j$. Since

$$\partial_{aJ}A_{bK} = i\int d^4\kappa\,\pi_a\bar\Pi_J\bar\Pi_K e^{i\pi^c\bar\Pi^L Y_{cL}}\Phi_b(\pi,\bar\Pi)\,,$$

A_{aJ} of (6.5) satisfies (5.2) where

$$F_{JK} = i\int d^4\kappa\,\bar\Pi_J\bar\Pi_K e^{i\pi^c\bar\Pi^L Y_{cL}}\pi^a\Phi_a(\pi,\bar\Pi)\,.$$

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So the onshell unintegrated vertex operator is

$$V = \lambda^a C^J A_{aJ}(Y) = \int d^4\kappa (C^J \bar{\Pi}_J)(\lambda^a \Phi_a(\pi, \bar{\Pi})) \exp(i\pi^b \bar{\Pi}^J Y_{bJ})\,. \tag{6.6}$$

To show that scattering amplitudes of these vertex operators satisfy the integrability properties of self-dual Yang-Mills, note that

$$Q(e^{i\pi^b \bar{\Pi}^J Y_{bJ}}) = i(\lambda^a \pi_a)(C^J \bar{\Pi}^J)(e^{i\pi^b \bar{\Pi}^J Y_{bJ}})$$

where $Q = \int dz (\lambda^a C^J \partial Y_{aJ} + \cdots)$. So V can formally be written as

$$V = Q[(\lambda^a \pi_a)^{-1}\Omega] \tag{6.7}$$

where

$$\Omega = -i \int d^4\kappa (\lambda^a \Phi_a(\pi, \bar{\Pi})) \exp(i\pi^b \bar{\Pi}^J Y_{bJ})\,.$$

Since $(\lambda^a \pi_a)^{-1}$ has singularities when λ^a is proportional to π^a, $(\lambda^a \pi_a)^{-1}$ is not globally defined and V is not BRST-trivial. However, the product of two onshell vertex operators can be expressed either as

$$V_1 V_2 = Q[(\lambda^a \pi_{1a})^{-1}\Omega_1 V_2] \qquad \text{or} \qquad V_1 V_2 = Q[(\lambda^a \pi_{2a})^{-1} V_1 \Omega_2]\,. \tag{6.8}$$

As long as $\pi_{1a}\pi_2^a$ is nonzero, either $(\lambda^a \pi_{1a})^{-1}$ or $(\lambda^a \pi_{2a})^{-1}$ is nonsingular. So whenever $\pi_{1a}\pi_2^a$ is nonzero, $V_1 V_2$ is BRST-trivial. This implies that N-point tree amplitudes vanish unless $\pi_r^a \pi_{sa} = 0$ for all $1 \leq r, s \leq N$, i.e. unless all external momenta are in the same "self-dual plane". This is consistent with the absence of massive states in the spectrum since $\pi_r^a \pi_{sa} = 0$ implies that all intermediate states are massless.

Unlike amplitudes computed using the $\mathcal{N} = 0$ version of the self-dual string, the amplitudes computed using this $\mathcal{N} = 4$ formalism are manifestly Lorentz covariant. This does not cause contradictions for loop amplitudes since all $\mathcal{N} = 4$ self-dual loop amplitudes are vanishing [16]. Furthermore, one can check that if one computes the field theory action

$$\mathcal{S} = \langle VQV + V^3 \rangle \tag{6.9}$$

using the normalization of (6.3) and the string field $V = \lambda^a C^J A_{aJ}(Y)$, one reproduces the self-dual super–Yang-Mills action of (5.5).

7. Conclusions and speculations

In these proceedings, an open string theory for $\mathcal{N} = 4$ $d = 4$ self-dual super–Yang-Mills was constructed which is related to the $d = 10$ pure spinor formalism of the superstring. Since the coupling constant of self-dual (super)-Yang-Mills can be eliminated by scaling G^{ab} and $a_{a\dot{a}}$ of (5.5) in opposite directions, self-dual (super) Yang-Mills is essentially a free theory. However, one can obtain the full (super) Yang-Mills action from the self-dual (super) Yang-Mills action of (5.5) by adding the perturbation term

$$S_{\text{pert}} = g_{\text{YM}} \int d^4x G^{ab} G_{ab} \,(+\text{supersymmetric terms}) \tag{7.1}$$

where $g_{\rm YM}$ is the Yang-Mills coupling constant and G^{ab} is the anti-self-dual field strength of (5.4). It is therefore important to understand the perturbation of (7.1) which turns self-dual (super) Yang-Mills into ordinary (super) Yang-Mills.[4]

Recently, it has been understood how to describe this perturbation using string theories in twistor space [9]. It would therefore be very interesting to relate these string theories in twistor space with the string theory in (x, θ) space which has been described here. Such a relationship would not be surprising since the pure spinor formalism for the superstring is closely related to theories in ten-dimensional twistor space [21]. Furthermore, equations (6.4)–(6.8) for vertex operators and scattering amplitudes in this string theory resemble twistor constructions involving the Penrose twistor-transform.

Acknowledgments

I would especially like to thank Edward Witten for many useful ideas and comments and for his collaboration during various stages of this project. I would also like to thank Rajesh Gopakumar, Warren Siegel and Cumrun Vafa for useful discussions, CNPq grant 300256/94-9, Pronex grant 66.2002/1998-9, and FAPESP grant 99/12763-0 for partial financial support, the Institute for Advanced Study for their hospitality, and the organizers of Strings 2003 for their excellent work.

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[4] At the Strings 2003 talk [8], it was speculated that this perturbation might be related to deforming the R^4 background to an AdS_4 background. In an AdS_4 background, the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{OSp}(4|2)$ rotations of Y_{aJ} combine with the $R^{4|8}$ translations of Y_{aJ} to form a supergroup $\mathrm{OSp}(4|4)$. Since the $\mathrm{OSp}(4|4)$ supergroup treats chiral and antichiral spinor indices in a symmetric manner, it was hoped that the string theory in this AdS_4 background might describe the full $\mathcal{N}=4$ super–Yang-Mills theory. However, there is a simple argument which makes this speculation unlikely. If one performs an $O(4)$ R-symmetry transformation which is not in the SO(4) subgroup of $\mathrm{OSp}(4|2)$, the self-dual super–Yang-Mills action of (5.5) transforms with opposite sign from the $g_{\rm YM} \int d^4x G^{ab} G_{ab}$ perturbation term. But since the term in the worldsheet action which deforms R^4 to AdS_4 is invariant under these $O(4)$ transformations, it cannot be responsible for this change in sign.

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